

Level - 3

JEE Advanced++ Pattern/Solutions

153. $\left(\frac{Lms}{KA} \ln \frac{\theta_0 - \theta_1}{\theta_0 - \theta_2} \right)$

Suppose, the temperature of the water in the smaller vessel is θ at time t . In the next time interval dt , a heat ΔQ is transferred to it where

$$dQ = \frac{KA}{L}(\theta_0 - \theta)dt. \quad \dots (i)$$

This heat increases the temperature of the water of mass m to $\theta + d\theta$ where

$$dQ = ms d\theta. \quad \dots (ii)$$

From (i) and (ii),

$$\frac{KA}{L}(\theta_0 - \theta)dt = ms d\theta \quad \text{or} \quad dt = \frac{Lms}{KA} \frac{d\theta}{\theta_0 - \theta} \quad \text{or} \quad \int_0^T dt = \frac{Lms}{KA} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\theta_0 - \theta}$$

where T is the time required for the temperature of the water to become θ_2 .

$$\text{Thus, } T = \frac{Lms}{KA} \ln \frac{\theta_0 - \theta_1}{\theta_0 - \theta_2}.$$

154. $\left[T = T_0 e^{-\frac{KA(C_1+C_2)}{C_1C_2}t} \right]$

Let T be the temperature difference between two blocks at time t .

Heat transferred per second,

$$\frac{dQ}{dt} = \frac{KAT}{l} \quad \dots (1)$$

Due to transfer of heat, temperature of one block is lowered while it increases for the other. Hence change in temperature difference,

$$dT = dT_1 + dT_2 \quad \dots (2)$$

Heat lost by one block is equal to the heat gained by the other.

$$C_1 dT_1 = C_2 dT_2 \quad \dots (3)$$

From eqns. (2) and (3),

$$dT = \left(\frac{C_1 + C_2}{C_2} \right) dT_1 \quad \dots (4)$$

If block one loses heat,

$$dQ = -C_1 dT_1$$

From eqn. (1),

$$\frac{dQ}{dt} = -C_1 \frac{dT_1}{dt} = \frac{KAT}{l} \quad \dots (5)$$

From eqns. (4) and (5),

$$-\frac{C_1 C_2}{C_1 + C_2} \frac{dT}{dt} = \frac{KA}{l} T ; \quad \frac{dT}{T} = -\frac{KA(C_1 + C_2)}{C_1 C_2} dt$$

$$\int_{T_0}^T \frac{dT}{T} = -\frac{KA(C_1 + C_2)}{C_1 C_2} t \quad \Rightarrow \quad T = T_0 e^{-\frac{KA(C_1 + C_2)}{C_1 C_2} t}$$

155. $\left(\frac{K}{4e\sigma L T_s^3 + K} \right)$

Rate of heat conduction through rod = rate of the heat lost from right end of the rod.

$$\therefore \frac{KA(T_1 - T_2)}{L} = eA\sigma(T_2^4 - T_s^4) \quad \dots (i)$$

Given that $T_2 = T_s + \Delta T$

$$\therefore T_2^4 = (T_s + \Delta T)^4 = T_s^4 \left(1 + \frac{\Delta T}{T_s} \right)^4$$

Using binomial expansion, we have

$$T_2^4 = T_s^4 \left(1 + 4 \frac{\Delta T}{T_s} \right) \quad (\text{as } \Delta T \ll T_s)$$

$$\therefore T_2^4 - T_s^4 = 4(\Delta T)(T_s^3)$$

Substituting in Eq. (i), we have $\frac{K(T_1 - T_s - \Delta T)}{L} = 4e\sigma T_s^3 \Delta T$

or $\frac{K(T_1 - T_s)}{L} = \left(4e\sigma T_s^3 + \frac{K}{L} \right) \Delta T \quad \therefore \Delta T = \frac{K(T_1 - T_s)}{(4e\sigma L T_s^3 + K)}$

Comparing with the given relation, proportionality constant = $\frac{K}{4e\sigma L T_s^3 + K}$

156. $\left(\frac{r\rho c}{9e\sigma} \left(\frac{1}{T_2^3} - \frac{1}{T_1^3} \right) \right)$

The rate of loss of energy due to radiation, $P = eA\sigma T^4$

This rate must be equal to $mc \frac{dT}{dt}$.

Hence, $-mc \frac{dT}{dt} = eA\sigma T^4$

Negative sign is used as temperature decreases with time. In this equation, $m = \left(\frac{4}{3} \pi r^3 \right) \rho$ and $A = 4\pi r^2$

$$\therefore -\frac{dT}{dt} = \frac{3e\sigma}{\rho cr} T^4 \quad \text{or,} \quad -\int_0^t dt = \frac{r\rho c}{3e\sigma} \int_{T_1}^{T_2} \frac{dT}{T^4}$$

Solving this, we get $t = \frac{r\rho c}{9e\sigma} \left(\frac{1}{T_2^3} - \frac{1}{T_1^3} \right)$.

157. $\left(\frac{\ln 2}{k} \right)$

We have, $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

Where θ_0 is the temperature of the surrounding and θ is the temperature of the body at time t . Suppose

$\theta = \theta_1$ at $t = 0$.

Then,

$$\int_{\theta_1}^{\theta} \frac{d\theta}{\theta - \theta_0} = -k \int_0^t dt \quad \text{or} \quad \ln \frac{\theta - \theta_0}{\theta_1 - \theta_0} = -kt$$

or $\theta - \theta_0 = (\theta_1 - \theta_0)e^{-kt}$.

The body continues to lose heat till its temperature becomes equal to that of the surrounding. The loss of heat in this entire period is $\Delta Q_m = ms(\theta_1 - \theta_0)$.

This is the maximum heat the body can lose. If the body loses half this heat, the decrease in its temperature will be, $\frac{\Delta Q_m}{2ms} = \frac{\theta_1 - \theta_0}{2}$.

If the body loses this heat in time t_1 , the temperature at t_1 will be $\theta_1 - \frac{\theta_1 - \theta_0}{2} = \frac{\theta_1 + \theta_0}{2}$.

Putting these values of time and temperature in (i), $\frac{\theta_1 + \theta_0}{2} - \theta_0 = (\theta_1 - \theta_0)e^{-kt_1}$

or $e^{-kt_1} = \frac{1}{2} \quad \text{or} \quad t_1 = \frac{\ln 2}{k}$.

158. $\left(300 + 12.5e^{\frac{-2KA t_1}{CL}} \right)$

In the first part of the question ($t \leq t_1$)

At $t = 0$, $T_X = T_0 = 400K$ and at $t = t_1$, $T_X = T_1 = 350K$

Temperature of atmosphere, $T_A = 300K$ (constant)

This cools down according to Newton's law of cooling.

Therefore, rate of cooling $\propto$ temperature difference.

$$\therefore \left(-\frac{dT}{dt} \right) = k(T - T_A)$$

$$\Rightarrow \frac{dT}{T - T_A} = -k dt \Rightarrow \int_{T_0}^{T_1} \frac{dT}{T - T_A} = -k \int_0^{t_1} dt$$

$$\Rightarrow \ln \left(\frac{T_1 - T_A}{T_0 - T_A} \right) = -kt_1 \Rightarrow kt_1 = -\ln \left(\frac{350 - 300}{400 - 300} \right)$$

$$\Rightarrow kt_1 = \ln(2) \quad \dots (i)$$

In the second part ($t > t_1$), body X cools by radiation

(according to Newton's law) as well as by conduction.

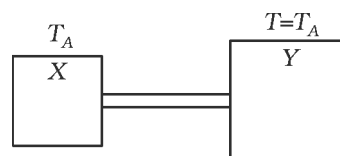
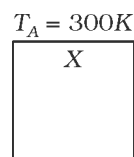
Therefore, rate of cooling = (cooling by radiation) + (cooling by conduction)

In conduction, $\frac{dQ}{dt} = \frac{KA(T - T_A)}{L} = C \left(-\frac{dT}{dt} \right) \therefore \left(-\frac{dT}{dt} \right) = \frac{KA}{LC}(T - T_A)$

where, C = heat capacity of body X

$$\therefore \left(-\frac{dT}{dt} \right) = k(T - T_A) + \frac{KA}{CL}(T - T_A) \dots (ii)$$

$$\left(-\frac{dT}{dt} \right) = \left(k + \frac{KA}{CL} \right) (T - T_A) \dots (iii)$$



Let at $t = 3t_1$, temperature of x becomes T_2

Then from Eq. (iii)

$$\int_{T_1}^{T_2} \frac{dT}{T - T_A} = - \left(k + \frac{KA}{KC} \right) \int_{t_1}^{3t_1} dt$$

$$\ln \left(\frac{T_2 - T_A}{T_1 - T_A} \right) = - \left(k + \frac{KA}{LC} \right) (2t_1) = - \left(2kt_1 + \frac{2KA}{LC} t_1 \right)$$

$$\text{or } \ln \left(\frac{T_2 - 300}{350 - 300} \right) = -2 \ln(2) - \frac{2KA t_1}{LC}; \quad kt_1 = \ln(2) \text{ from Eq. (i).}$$

$$\text{This equation gives } T_2 = \left(300 + 12.5 e^{\frac{-2KA t_1}{CL}} \right) K$$